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Nonlinear PDEs and measure-valued branching type processes

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ABSTRACT

We deal with the probabilistic approach to a nonlinear operator Λ of the form $\Lambda u = \Delta u + \sum_{k=1}^{\infty} q_k u^k$, in connection with the works of M. Nagasawa, N. Ikeda, S. Watanabe, and M.L. Silverstein on the discrete branching processes. Instead of the Laplace operator we may consider the generator of a right (Markov) process, called base process, with a general (not necessarily locally compact) state space. It turns out that solutions of the nonlinear equation $\Lambda u = 0$ are produced by the harmonic functions with respect to the (linear) generator of a discrete branching type process. The consideration of the general state space allows to take as base process a measure-valued superprocess (in the sense of E.B. Dynkin). The probabilistic counterpart is a Markov process which is a combination between a *continuous* branching process (e.g., associated with a nonlinear operator of the form $\Delta u - u^{\alpha}$, $1 < \alpha \leqslant 2$) and a *discrete* branching type one, on a space of configurations of finite measures. Our approach uses probabilistic and analytic potential theoretical tools, like the potential kernel of a continuous additive functional and the subordination operators.

1. Introduction

The classical works of M. Nagasawa, N. Ikeda, S. Watanabe, and M.L. Silverstein (cf. [19,15,16,24]) emphasized a natural connection between the discrete branching processes and nonlinear partial differential operators Λ of the type

$$\Lambda u = \Delta u + \sum_{k=1}^{\infty} q_k u^k,$$

on an open subset of a Euclidean space, where the coefficients q_k are positive, Borelian functions with $\sum_{k=1}^{\infty} q_k \leqslant 1$. More precisely, the semigroup of nonlinear operators generated by Λ is used to prove existence results for the branching processes. One can replace Δ with the generator of a standard (Markov) process (called base process) with state space a metrizable locally compact space. Furthermore, instead of $\sum_{k=1}^{\infty} q_k u^k$ it is possible to consider a more general nonlinear part for Λ, generated by a "branching" kernel.

The aim of this paper is to investigate operators Λ of the above type, by means of the associated branching type Markov processes, using potential theoretical tools, with applications to infinite dimensional situations. In particular, we show that solutions of the nonlinear equation

$$\Delta u + \sum_{k=1}^{\infty} q_k u^k = 0$$

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are produced by harmonic functions with respect to the generator of the associated branching type process. We complete results of M. Nagasawa from [20], where a similar probabilistic approach was used for solving the Dirichlet problem associated with the nonlinear operator Λ.

We avoid imposing secondary hypotheses (like the local compactness of the base space) which would limit the domain of applicability and therefore we can construct discrete branching type processes, starting from a continuous branching process (i.e., a superprocess in the sense of E.B. Dynkin) as base process. The generator of a superprocess is rather a second-order integro-differential operator than a differential one and the state space is the set of all finite measures on the initial space E. Consequently, it is crucial for the treatment of this application the consideration of a general state space in the above mentioned investigation of the discrete branching processes. The final result of the construction will be a Markov process on the finite configurations of positive finite measures on E.

The probabilistic counterpart of our approach is a revisit of the discrete branching process theory as developed by Ikeda–Nagasawa–Watanabe. Our main potential theoretical arguments are two successive perturbations of the resolvent family of a right (Markov) process. The first one is associated with a subordination operator in the sense of G. Mokobodzki and it corresponds to the killing of a process with a multiplicative functional. This necessary transformation is followed by an "inverse subordination" perturbation induced by a branching kernel and it corresponds to a "renaissance" of the process (in the sense of [18]).

The paper is organized as follows. The next two sections introduce our setup, following essentially [15]. We give the appropriate completion results, demanded by the consideration of a more general topological base space E (of Radon or Lusin type instead of a locally compact one). In Section 2 we collect some basic facts on the space S of all finite configurations of E (the proofs are presented in Appendix A), while Section 3 is devoted to the branching kernels. Appendix A includes as well complements on the resolvents of kernels and the associated right processes, the continuous additive functionals, and on the subordination operators and subordinate resolvents.

The main statements of the work are found in Section 4: perturbations induced by kernels (Proposition 4.5 and Corollary 4.6) and the application at the construction of the discrete branching type process generated by the base process and a pair of killing and branching kernels (Theorem 4.10), solutions of the nonlinear equation of the form $\Lambda u = 0$, produced by the harmonic functions with respect to the branching type processes (Corollary 4.13 and Corollary 4.15).

In Section 5 we present the claimed application to the study of a combination between a continuous branching process and a discrete branching one.

Section 6 is an addendum, exposing the treatment of the Dirichlet problem associated with the nonlinear operator Λ. Following the approach of M.L. Silverstein [24], we first outline the construction of a branching semigroup on S by solving an appropriate integral equation (Proposition 6.1). As in [20], it turns out that the restriction to E of an invariant function with respect to this semigroup is a solution of the Dirichlet problem associated with Λ (Proposition 6.4).

The obtained results are completed by remarks on various connections with the classical works on branching processes.

2. The general framework

Let $(E, \mathcal{T})$ be a Radon topological space (i.e., it is homeomorphic to a universally measurable subset of a compact metrizable space), denote by $\mathcal{B} = \mathcal{B}(E)$ its Borel σ-algebra and by $p\mathcal{B}$ the set of all positive, numerical, $\mathcal{B}$-measurable functions.

For every natural number m we consider the m times symmetric power $E^{(m)}$ of E, i.e., the factorization of the Cartesian product E^m by the equivalence relation induced by the permutation group σ^m: $x, y \in E^m$ are equivalent if and only if there exists a permutation $\tau \in \sigma^m$ such that $y = \overline{\tau}(x)$, where $\overline{\tau} : E^m \longrightarrow E^m$ is defined as $\overline{\tau}(x_1, \dots, x_m) := (x_{\tau(1)}, \dots, x_{\tau(m)})$ for $(x_1, \dots, x_m) \in E^m$. The space $E^{(m)}$ is endowed with the quotient topology $\mathcal{T}^{(m)}$, where E^m is equipped with the product topology $\mathcal{T}^m$.

A real-valued function g defined on E^m is called *invariant* if $g(x) = g(y)$ for all $y \in \widetilde{x}$, where $\widetilde{x}$ denotes the equivalence class of $x \in E^m$. For a function $f : E^m \longrightarrow \mathbb{R}$ we may consider the invariant function $f^{\#} : E^m \longrightarrow \mathbb{R}$ defined as

$$f^{\#}(x) := \frac{1}{m!} \sum_{\tau \in \sigma^m} f\big(\overline{\tau}(x)\big), \quad x \in E^m.$$

Clearly, $f^{\#} = f$ if and only if f is invariant. We define the projection $\widetilde{f} : E^{(m)} \longrightarrow \mathbb{R}$ of f on $E^{(m)}$ by

$$\widetilde{f}(\widetilde{x}) := f^{\#}(x), \quad \widetilde{x} \in E^{(m)}.$$

We denote by $\mathcal{B}^m$ (resp. $\mathcal{B}^{(m)}$) the Borel σ-algebra on E^m (resp. on $E^{(m)}$).

A kernel N on $(E^m, \mathcal{B}^m)$ is called *invariant* provided that the function Nf is invariant for every $f \in p\mathcal{B}^m$ which is invariant.

Remark 2.1. The following assertions hold for a kernel N on $(E^m, \mathcal{B}^m)$.

(i) N is invariant if and only if $N(f^{\#})^{\#} = N(f^{\#})$ for all $f \in p\mathcal{B}^m$.
(ii) If $(Nf)^{\#} = N(f^{\#})$ for all $f \in p\mathcal{B}^m$ then N is invariant.

A useful example of invariant kernel is the m times product of a kernel on E. More precisely, let N be a kernel on $(E,\mathcal{B})$. Then for every $m \geqslant 2$ we consider the kernel N^m on $(E^m,\mathcal{B}^m)$ defined as

$$N^m f(x) := \underbrace{\int \cdots \int}_{m \text{ times}} f(y_1,\ldots,y_m) N(x_1,dy_1)\cdots N(x_m,dy_m)$$

for all $f \in p\mathcal{B}^m$ and $x=(x_1,\ldots,x_m) \in E^m$.

The following three lemmas collect several basic measure theoretical and topological properties of the spaces $E^{(m)}$, $m \geqslant 2$. For the reader convenience we present their proofs in Appendix A.2 of the paper.

Lemma 2.2. *Let N be a kernel on $(E,\mathcal{B})$. If $m \geqslant 2$ then*

$$\left(N^m f\right)^{\#} = N^m\left(f^{\#}\right) \quad \text{for all } f \in p\mathcal{B}^m.$$

In particular, the kernel N^m is invariant. If N is sub-Markovian (i.e., $N1 \leqslant 1$) then N^m is sub-Markovian too.

The state space for the forthcoming branching process will be the set S of all positive measures μ on E which are finite sums of Dirac measures: $\mu = \sum_{k=1}^m \delta_{x_k}$, where $x_1,\ldots,x_m \in E$, called (cf. [24]) the space of *finite configurations* of E. S is identified (see, e.g., [15]) with the direct sum of all symmetric m-th powers $E^{(m)}$ of E, hence

$$S = \bigoplus_{m\geqslant 1} E^{(m)},$$

and it is equipped with the canonical topological structure. We denote by $\mathcal{B}(S)$ the Borel σ-algebra on S.

Recall that $(E,\mathcal{T})$ is a *Lusin topological space* if it is homeomorphic to a Borel subset of a compact metrizable space.

Lemma 2.3. *If $(E,\mathcal{T})$ is compact metrizable then $E^{(m)}$ is also compact metrizable. If E is a Lusin (resp. Radon) topological space then $(E^{(m)},\mathcal{T}^{(m)})$ and S are of the same type.*

For every real-valued, $\mathcal{B}$-measurable function φ consider the function $\widehat{\varphi} : \bigoplus_{m\geqslant 1} E^m \longrightarrow \mathbb{R}$ defined as

$$\widehat{\varphi}(x) := \varphi(x_1)\cdot\cdots\cdot\varphi(x_m) \quad \text{for } x=(x_1,\ldots,x_m) \in E^m.$$

Note that the restriction of $\widehat{\varphi}$ to each E^m is invariant and therefore its projection on S, $\widetilde{\widehat{\varphi}} : S \longrightarrow \mathbb{R}$, is precisely

$$\widetilde{\widehat{\varphi}}(\widetilde{x}) = \widehat{\varphi}(x) = \varphi(x_1)\cdot\cdots\cdot\varphi(x_m),$$

where $m \geqslant 1$ and $\widetilde{x} \in E^{(m)}$, $x=(x_1,\ldots,x_m) \in E^m$.

In what follows, to simplify the notations, we also denote by $\widehat{\varphi}$ the projection $\widetilde{\widehat{\varphi}}$ of $\widehat{\varphi}$ on S. Such a function $\widehat{\varphi}$ is called *multiplicative* (cf. [24]).

We denoted by $bp\mathcal{B}$ the bounded elements of $p\mathcal{B}$. For a class of functions $\mathcal{F} \subset bp\mathcal{B}$ we denote by $\mathcal{F}^{\otimes m}$ the family of functions on E^m defined as

$$\mathcal{F}^{\otimes m} = \left\{ f_1 \otimes \cdots \otimes f_m \;\middle|\; f_i \in \mathcal{F},\ i \in \{1,\ldots,m\}\right\}.$$

Assertion (iii) of the following lemma is a measure theoretical version of a result from [15] on the continuous functions on S.

Lemma 2.4. *The following assertions hold.*

(i) *We have $p\mathcal{B}^{(m)} = \{\widetilde{f} : E^{(m)} \longrightarrow \overline{\mathbb{R}}_+ \mid f \in p\mathcal{B}^m\}$.*
(ii) *The Borel σ-algebra of $E^{(m)}$ is generated by $\widetilde{(bp\mathcal{B})^{\otimes m}} := \{\widetilde{f} \mid f \in (bp\mathcal{B})^{\otimes m}\}$.*
(iii) *Let $\mathcal{A} := \{\widehat{\varphi} \mid \varphi \in p\mathcal{B},\ \varphi \leqslant 1\}$, then $\sigma(\mathcal{A}) = \mathcal{B}(S)$.*
(iv) *If μ_1 and μ_2 are two finite measures on $(S,\mathcal{B}(S))$ such that $\mu_1(\widehat{\varphi}) = \mu_2(\widehat{\varphi})$ for all $\varphi \in p\mathcal{B}$, $\varphi \leqslant 1$, then $\mu_1 = \mu_2$.*

Let $M(E)$ be the set of all positive finite measures on E. For every $f \in p\mathcal{B}$ consider the mappings $l_f : M(E) \longrightarrow \overline{\mathbb{R}}_+$ and $e_f : M(E) \longrightarrow [0,1]$, defined by

$$l_f(\mu) := \langle \mu, f\rangle := \int f\,d\mu, \quad \mu \in M(E), \qquad e_f := \exp(-l_f).$$

The Borel σ-algebra $\mathcal{M}(E)$ on $M(E)$ is generated by $\{l_f \mid f \in bp\mathcal{B}\}$.

With this notation a multiplicative function $\widehat{\varphi}$, $\varphi \in p\mathcal{B}$, $\varphi \leqslant 1$, is the restriction to S of an exponential function on $M(E)$,

$$\widehat{\varphi} = e_{-\ln\varphi}.$$

Recall that if p_1, p_2 are two finite measures on $M(E)$, then their convolution $p_1 * p_2$ is the finite measure on $M(E)$ defined for every $F \in bp\mathcal{M}(E)$ by

$$\int p_1 * p_2(d\nu)F(\nu) := \int p_1(d\nu_1)\int p_2(d\nu_2)F(\nu_1+\nu_2).$$

In particular, if $f \in p\mathcal{B}$ then

$$p_1 * p_2(e_f) = p_1(e_f)p_2(e_f). \tag{2.1}$$

Note that if p_1 and p_2 are concentrated on S then $p_1 * p_2$ has the same property. If $\widehat{\varphi}$ is a multiplicative function on S, then $\widehat{\varphi}(\mu+\nu) = \widehat{\varphi}(\mu)\widehat{\varphi}(\nu)$ for all $\mu, \nu \in S$ and therefore

$$p_1 * p_2(\widehat{\varphi}) = p_1(\widehat{\varphi})p_2(\widehat{\varphi}). \tag{2.2}$$

3. Branching kernels

A kernel $N : p\mathcal{B}(S) \longrightarrow p\mathcal{B}(S)$ on the measurable space $(S, \mathcal{B}(S))$ can be identified through a matrix of kernels $(N_{ij})_{i,j\geqslant 1}$, where N_{ij} is a kernel from $(E^{(i)}, \mathcal{B}(E^{(i)}))$ to $(E^{(j)}, \mathcal{B}(E^{(j)}))$,

$$Nf|_{E^{(i)}} = \sum_{j\geqslant 1} N_{ji}(f|_{E^{(j)}}), \quad f \in p\mathcal{B}(S),\ i \geqslant 1.$$

The kernel N on $(S, \mathcal{B}(S))$ is called *diagonal* if $N_{ij} = 0$ for all $i, j \geqslant 1$, $i \neq j$. In this case we write

$$N = \bigoplus_{i\geqslant 1} N_{ii}.$$

According with [24], a kernel N on $(S, \mathcal{B}(S))$ (resp. on $(M(E), \mathcal{M}(E))$) which is sub-Markovian (i.e., $N1 \leqslant 1$) is called *branching kernel* provided that for all $\mu, \nu \in S$ (resp. for all $\mu, \nu \in M(E)$) we have

$$N_{\mu+\nu} = N_\mu * N_\nu, \tag{3.1}$$

where N_μ denotes the measure on $(S, \mathcal{B}(S))$ (resp. on $(M(E), \mathcal{M}(E))$) such that $Ng(\mu) = \int g\, dN_\mu$ for all $g \in bp\mathcal{B}(S)$ (resp. $g \in bp\mathcal{M}(E)$).

The following remark shows that the definition of branching kernel on S agrees with that considered by M. Nagasawa in [15].

Remark 3.1. Let N be a sub-Markovian kernel on $(S, \mathcal{B}(S))$. Then the following assertions are equivalent.

(i) N is a branching kernel.
(ii) For all $\varphi \in p\mathcal{B}$, $\varphi \leqslant 1$,

$$N\widehat{\varphi} = \widehat{(N\widehat{\varphi})|_E}.$$

(iii) N maps multiplicative functions into multiplicative functions.

The equivalence (i) $\Longleftrightarrow$ (ii) *can be easily verified, observing that by assertion* (iv) *of Lemma* 2.4 *to check* (3.1) *it is sufficient to consider multiplicative functions.*

Proposition 3.2.

(i) *For every sub-Markovian kernel $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ there exists a branching kernel $\widehat{B}$ on $(S, \mathcal{B}(S))$ such that for every $\mathcal{B}$-measurable function φ, $|\varphi| \leqslant 1$, we have*

$$\widehat{B}\widehat{\varphi} = \widehat{B\widehat{\varphi}}.$$

(ii) *Conversely, if H is a branching kernel on $(S, \mathcal{B}(S))$ then there exists a unique sub-Markovian kernel $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ such that $H = \widehat{B}$.*

Proof. (i) For every $x \in E$ we consider the measure B_x on $(S, \mathcal{B}(S))$,

$$B_x(g) := Bg(x) \quad \text{for all } g \in bp\mathcal{B}(S).$$

As in the proof of Lemma 2.1 from [24], if $\mu \in S$, $\mu = \delta_{x_1} + \cdots + \delta_{x_m}$, we define the measure $\widehat{B}_\mu$ on $(S, \mathcal{B}(S))$ by convolution,

$$\widehat{B}_\mu := B_{x_1} * \cdots * B_{x_m}.$$

The function $\widehat{B}g$ on S, $g \in bp\mathcal{B}(S)$, is then defined by

$$\widehat{B}g(\mu) := \widehat{B}_\mu(g) \quad \text{for all } \mu \in S.$$

Using (2.2) we have $\widehat{B}\widehat{\varphi}(\mu) = \widehat{B}_\mu(\widehat{\varphi}) = \prod_{i=1}^m B\widehat{\varphi}(x_i) = \widehat{B\widehat{\varphi}}(\mu)$. In particular, the function $\widehat{B}g$ belongs to $p\mathcal{B}(S)$ if $g = \widehat{\varphi}$, $\varphi \leqslant 1$. By assertion (iii) of Lemma 2.4 and using a monotone class argument, we conclude that $\widehat{B}g \in p\mathcal{B}(S)$ for all $g \in bp\mathcal{B}(S)$. Therefore $\widehat{B}$ is a kernel on $(S, \mathcal{B}(S))$.

(ii) Defining the kernel B by $Bg := Hg|_E$, $g \in \mathcal{B}(S)$, it is easy to check that $\widehat{B}\widehat{\varphi} = H\widehat{\varphi}$ for all $\varphi \in p\mathcal{B}$, $\varphi \leqslant 1$. The assertion holds now by Lemma 2.4(iv). $\square$

Example of branching kernel. The standard example of branching kernel is $\widehat{B}$ given by Proposition 3.2, where

$$Bg(x) := \sum_{k \geqslant 1} q_k(x) g_k(x, \ldots, x), \quad g \in bp\mathcal{B}(S),\ x \in E, \tag{3.2}$$

with $g_k := g|_{E^{(k)}}$, $q_k \in p\mathcal{B}$ for all $k \geqslant 1$, satisfying $\sum_{k \geqslant 1} q_k \leqslant 1$. In particular, for all $\varphi \in p\mathcal{B}$, $\varphi \leqslant 1$, we have

$$B\widehat{\varphi} = \sum_{k \geqslant 1} q_k \varphi^k.$$

4. Markov processes on the finite configurations

We assume further that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a right (Markov) process with state space the Radon topological space E; see, e.g., [23] and [2] for details. X is called further *base process*.

We denote by $(P_t)_{t \geqslant 0}$ the transition function and by $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ the resolvent of X,

$$P_t f(x) = E^x(f \circ X_t), \qquad U_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\, dt, \quad x \in E,\ f \in p\mathcal{B}.$$

A positive numerical universally $\mathcal{B}$-measurable function u on E is called *$\mathcal{U}$-excessive* provided that $\alpha U_\alpha u \leqslant u$ for all $\alpha > 0$ and $\lim_{\alpha \to \infty} \alpha U_\alpha u(x) = u(x)$ for all $x \in E$. We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}$-measurable $\mathcal{U}$-excessive functions.

If $\beta > 0$ we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $(U_{\beta+\alpha})_{\alpha > 0}$.

4.1. Markov processes on the symmetric m-th power

For $m \geqslant 2$, let $X^m = (\Omega^m, \mathcal{F}^m, \mathcal{F}_t^m, X_t^m, \theta_t^m, P^x)$ be the m times Cartesian power of $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$: $\Omega^m := \underbrace{\Omega \times \cdots \times \Omega}_{m \text{ times}}$, $X_t^m := (\underbrace{X_t, \ldots, X_t}_{m \text{ times}})$, $\theta_t^m(\omega_1, \ldots, \omega_m) := (\theta_t \omega_1, \ldots, \theta_t \omega_m)$ for $(\omega_1, \ldots, \omega_m) \in \Omega^m$, $P^{(x_1, \ldots, x_m)} = P^{x_1} \otimes \cdots \otimes P^{x_m}$ for $(x_1, \ldots, x_m) \in E^m$.

By Theorem (15.2) in [23] we deduce that X^m is a right process with state space E^m and its transition function is

$$P_t^m = \underbrace{P_t \otimes \cdots \otimes P_t}_{m \text{ times}}, \quad t \geqslant 0.$$

Let $\mathcal{U}^m = (U_\alpha^m)_{\alpha > 0}$ be the resolvent family of X^m, i.e., the resolvent of kernels on $(E^m, \mathcal{B}^m)$ induced by the semigroup $(P_t^m)_{t \geqslant 0}$,

$$U_\alpha^m f = \int_0^\infty e^{-\alpha t} P_t^m f\, dt, \quad f \in p\mathcal{B}^m.$$

By Lemma 2.2 we get that P_t^m is invariant for all $t \geqslant 0$, therefore U_α^m is also invariant for all $\alpha > 0$ and

$$U_\alpha^m\big(f^\#\big) = \big(U_\alpha^m f\big)^\#, \quad f \in p\mathcal{B}^m. \tag{4.1}$$

For a kernel N on E^m we may consider its projection $\widetilde{N}$ on $(E^{(m)}, \mathcal{B}^{(m)})$ defined as

$$\widetilde{N}\widetilde{h} := N\widetilde{(\widetilde{h} \circ \pi)}, \quad \widetilde{h} \in p\mathcal{B}^{(m)},$$

where π is the canonical projection, $\pi : E^m \longrightarrow E^{(m)}$, $\pi(x) := \widetilde{x}$. Note that by assertion (i) of Lemma 2.4 $\widetilde{N}\widetilde{h} \in p\mathcal{B}^{(m)}$. Hence we may consider the projections $P_t^{(m)}$, $t \geqslant 0$, and $U_\alpha^{(m)}$, $\alpha > 0$, of P_t^m and U_α^m respectively on $(E^{(m)}, \mathcal{B}^{(m)})$,

$$P_t^{(m)} = \widetilde{P_t^m}, \qquad U_\alpha^{(m)} = \widetilde{U_\alpha^m}.$$

Let us set $\mathcal{U}^{(m)} = (U_\alpha^{(m)})_{\alpha>0}$.

If $M \in \mathcal{B}$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$, then the *reduced function* (with respect to $\mathcal{U}_\beta$) *of* u *on* M is the function $R_\beta^M u$ defined by

$$R_\beta^M u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) \mid v \geqslant u \text{ on } M\}.$$

The reduced function $R_\beta^M u$ is universally $\mathcal{B}$-measurable, it is supermedian with respect to $\mathcal{U}_\beta$ (i.e., $\alpha U_{\beta+\alpha}(R_\beta^M u) \leqslant R_\beta^M u$ for all $\alpha > 0$), and by Hunt's Theorem we have $R_\beta^M u(x) = E^x[e^{-\beta D_M} u(X_{D_M})]$, $x \in E$, where D_M is the *entry time* of M, $D_M = \inf\{t \geqslant 0 \mid X_t \in M\}$.

Proposition 4.1. *The following assertions hold.*

(i) *The family $(P_t^{(m)})_{t\geqslant 0}$ is a sub-Markovian semigroup of kernels on $(E^{(m)}, \mathcal{B}^{(m)})$ and the induced resolvent of kernels is the family $\mathcal{U}^{(m)} = (U_\alpha^{(m)})_{\alpha>0}$.*

(ii) *If $\beta > 0$ and $\widetilde{h} \in p\mathcal{B}^{(m)}$ then $\widetilde{h} \in \mathcal{E}(\mathcal{U}_\beta^{(m)})$ (resp. $\widetilde{h}$ is $\mathcal{U}_\beta^{(m)}$-supermedian) if and only if $\widetilde{h} \circ \pi \in \mathcal{E}(\mathcal{U}_\beta^m)$ (resp. $\widetilde{h} \circ \pi$ is $\mathcal{U}_\beta^m$-supermedian). The map $\widetilde{h} \longmapsto \widetilde{h} \circ \pi$ is an order preserving bijection between the convex cone $\mathcal{E}(\mathcal{U}_\beta^{(m)})$ and the invariant elements of $\mathcal{E}(\mathcal{U}_\beta^m)$.*

(iii) *If $u \in \mathcal{E}(\mathcal{U}_\beta^m)$ then $u^\# \in \mathcal{E}(\mathcal{U}_\beta^m)$. If $p \in \mathcal{E}(\mathcal{U}_\beta^m)$ is invariant and $F \in \mathcal{B}^m$, $F = F^\#$, then $\widetilde{R}_\beta^{\widetilde{F}} \widetilde{p} = \widetilde{R_\beta^F p}$, where $F^\# := \bigcup_{x\in F} \widetilde{x}$, $\widetilde{F} := \pi(F^\#)$ and $\widetilde{R}_\beta^{\widetilde{F}} \widetilde{p}$ denotes the reduced function (with respect to $\mathcal{U}_\beta^{(m)}$) of $\widetilde{p}$ on $\widetilde{F}$.*

Proof. (i) If $s, t > 0$ and $\widetilde{h} \in p\mathcal{B}^{(m)}$ then $P_t^{(m)}(P_s^{(m)}\widetilde{h}) = P_t^{(m)}(P_s^m\widetilde{(\widetilde{h} \circ \pi)}) = P_t^m\widetilde{(P_s^m(\widetilde{h} \circ \pi))} = P_{t+s}^{(m)}\widetilde{h}$. We have also $P_t^{(m)}1 = \widetilde{P_t^m 1} \leqslant 1$. The proof of the last part of (i) is straightforward.

Assertion (ii) holds because $U_\alpha^{(m)}\widetilde{h} = U_\alpha^m\widetilde{(\widetilde{h} \circ \pi)}$.

(iii) If $u \in \mathcal{E}(\mathcal{U}_\beta^m)$ then there exists a sequence $(f_k)_k \subset p\mathcal{B}^m$ such that $U_\beta^m f_k \nearrow u$ and therefore $U_\beta^m(f_k^\#) = (U_\beta^m f_k)^\# \nearrow u^\#$. We conclude that $u^\# \in \mathcal{E}(\mathcal{U}_\beta^m)$. Let now $p \in \mathcal{E}(\mathcal{U}_\beta^m)$, $p = p^\#$ and $F \in \mathcal{B}^m$, $F = F^\#$. Then by (ii) we have $\widetilde{R}_\beta^{\widetilde{F}}\widetilde{p} = \inf\{\widetilde{v} \mid v \in \mathcal{E}(\mathcal{U}_\beta^m),\ v = v^\#,\ \widetilde{v} \geqslant \widetilde{p} \text{ on } \widetilde{F}\} = \inf\{\widetilde{v} \mid v \in \mathcal{E}(\mathcal{U}_\beta^m),\ v^\# \geqslant p \text{ on } F\} \leqslant \inf\{\widetilde{v} \mid v \in \mathcal{E}(\mathcal{U}_\beta^m),\ v \geqslant p \text{ on } F\} = \widetilde{R_\beta^F p}$. Conversely, if $v \in \mathcal{E}(\mathcal{U}_\beta^m)$, $v^\# \geqslant p$ on F then $v^\# \geqslant R^F p$, $v^\# \geqslant (R_\beta^F p)^\#$, and thus $\widetilde{v} \geqslant \widetilde{R_\beta^F p}$, $\widetilde{R}_\beta^{\widetilde{F}}\widetilde{p} \geqslant \widetilde{R_\beta^F p}$. $\square$

We suppose in the sequel that E is a Lusin topological space.

Proposition 4.2. *The semigroup of kernels $(P_t^{(m)})_{t\geqslant 0}$ is the transition function of a right (Markov) process with state space $E^{(m)}$.*

Proof. Since by Lemma 2.3 $E^{(m)}$ is a Lusin topological space, according with Appendix A.1 we have to show that the associated resolvent $\mathcal{U}^{(m)} = (U_\alpha^{(m)})_{\alpha>0}$ satisfies for one $\beta > 0$ the property $(A.1b)$. By assertion (ii) of Proposition 4.1 is clear that $1 \in \mathcal{E}(\mathcal{U}_\beta^{(m)})$ and that $\mathcal{E}(\mathcal{U}_\beta^{(m)})$ is min-stable. Note that since $(P_t^m)_{t\geqslant 0}$ is the transition function of a right process, it follows that for every $f \in b\mathcal{C}(E^m)$ and $x \in E^m$ we have $\lim_{t\longrightarrow 0} P_t^m f(x) = f(x)$. Consequently we get $\lim_{\alpha\longrightarrow\infty} \alpha U_\alpha^m f(x) = f(x)$ and therefore $\lim_{\alpha\longrightarrow\infty} \alpha U_\alpha^{(m)}\widetilde{h}(\widetilde{x}) = \widetilde{h}(\widetilde{x})$ for all $\widetilde{h} \in b\mathcal{C}(E^{(m)})$. We conclude that every bounded continuous function on $E^{(m)}$ is $\sigma(\mathcal{E}(\mathcal{U}_\beta^{(m)}))$-measurable and so $\sigma(\mathcal{E}(\mathcal{U}_\beta^{(m)})) = \mathcal{B}^{(m)}$. For a σ-finite measure ν on $(E^m, \mathcal{B}^m)$ we put

$$\nu^\# := \nu \circ i \quad \text{and} \quad \widetilde{\nu} := \nu \circ \pi^{-1},$$

where i is the kernel on E^m defined as $i(f) = f^\#$. Clearly $\nu^\#$ (resp. $\widetilde{\nu}$) is a σ-finite measure on $(E^m, \mathcal{B}^m)$ (resp. $(E^{(m)}, \mathcal{B}^{(m)})$) and $\widetilde{\nu^\#} = \widetilde{\nu}$. If μ is a σ-finite measure on $(E^{(m)}, \mathcal{B}^{(m)})$ then we consider the measure μ_0 on $(E^m, \mathcal{B}^m)$ defined as $\mu_0(f) := \mu(\widetilde{f})$, $f \in p\mathcal{B}^m$. We have

$$\mu_0 = (\mu_0)^\# \quad \text{and} \quad \widetilde{\mu_0} = \mu.$$

If $\eta \in Exc(\mathcal{U}^m_\beta)$ then $\widetilde{\eta} \in Exc(\mathcal{U}^{(m)}_\beta)$ and if in addition $\eta = \nu \circ U^m_\beta$ then $\widetilde{\eta} = \widetilde{\nu} \circ U^{(m)}_\beta$. Conversely, if $\xi \in Exc(\mathcal{U}^{(m)}_\beta)$ then using (4.1) one can see that $\xi_0 \in Exc(\mathcal{U}^m_\beta)$ and if $\xi = \mu \circ U^{(m)}_\beta$ then $\xi_0 = \mu_0 \circ U^m_\beta$. From the above consideration it is easy to check that the second assertion of $(A.1b)$ holds.

Let $p_0 := U^m_\beta 1$, G be an open subset of E^m such that $G = G^\#$, and put $F = E^m \setminus G$. Recall that F is finely closed (with respect to $\mathcal{U}^m_\beta$) if and only if it is thin in any point of G, i.e., $R^F_\beta p_0(x) < p_0(x)$ for all $x \in G$. Let $\widetilde{x} \in \widetilde{G}$. Since G is finely open with respect to $\mathcal{U}^m_\beta$ and $x \in G$, the above inequality holds. We have $F = F^\#$ and by (4.1), $p_0^\# = p_0$. Because $R^F_\beta p_0(x) < p_0(x)$ we deduce that $(R^F_\beta p_0)^\#(x) < p_0(x)$ and by Proposition 4.1(iii) $\widetilde{R}^{\widetilde{F}}_\beta \widetilde{p_0}(\widetilde{x}) = \widetilde{R^F_\beta p_0}(\widetilde{x}) = (R^F_\beta p_0)^\#(x) < p_0(x) = \widetilde{p_0}(\widetilde{x})$. We conclude that $\widetilde{F}$ is thin in $\widetilde{x} \in \widetilde{G}$, hence $\widetilde{F}$ is finely closed, i.e., the last assertion of $(A.1b)$ holds. □

Remark 4.3. In the particular case when E is compact, the right (Markov) process given by Proposition 4.2 should be compared with the one constructed in [15, Section 1.2]; see also Proposition 4.1 from [19].

4.2. Perturbations induced by branching kernels

We present now several results on the perturbation induced by potential kernels; see Appendix A.1, Section 5.1 from [2], and [4] for details.

Let $q > 0$ and K^q be a (proper) *regular excessive* kernel with respect to $\mathcal{U}_q$, i.e., $K^q f$ is a $\mathcal{U}_q$-excessive function for every $f \in p\mathcal{B}$ and if $v \in \mathcal{E}(\mathcal{U}_q)$ is such that $K^q f \leqslant v$ on the set $[f > 0]$ then the inequality holds on E.

We assume that K^q is the initial kernel of a sub-Markovian resolvent of kernels $(K^q_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that $\inf_\alpha \alpha K^q_\alpha(U_q 1) = 0$. Note that the above condition holds if $K^q 1 < \infty$ and it implies

$$\inf_k \big(K^q_\beta\big)^k U_q 1 = 0. \tag{4.2}$$

By Theorem 1.3 from [4] it follows K^q_1 is a subordination operator with respect to $\mathcal{U}_q$ such that $E_{K^q_1} = E$. Let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be the subordinate resolvent generated by K^q_1. If $W = \sup_\alpha W_\alpha$ is the initial kernel of $\mathcal{W}$ then

$$W = U_q - K^q_1 U_q, \quad U_q = W + \sum_{k=1}^{\infty} \big(K^q_1\big)^k W.$$

The second equality holds by Proposition 5.1.26 from [2], since by (4.2) we get $\inf_k (K^q_1)^k U_q 1 = 0$.

A typical example of kernel K^q is given by the q-potential kernel U^q_A of a continuous additive functional $A = (A_t)_{t\geqslant 0}$ of the process X; see Appendix A.1, [23], and also [2] for the transient case. In this case if $f \in p\mathcal{B}$, $x \in E$, and $q > 0$ we have

$$K^q f(x) = U^q_A f(x) = E^x \int_0^\infty e^{-qt} f(X_t)\, dA_t,$$

$$K^q_\alpha f(x) = E^x \int_0^\infty e^{-qt-\alpha A_t} f(X_t)\, dA_t, \qquad W_\alpha f(x) = E^x \int_0^\infty e^{-qt-\alpha A_t} f(X_t)\, dt.$$

Remark 4.4.

(i) The family $\mathcal{W} = (W_\alpha)_{\alpha>0}$ is the resolvent of a right (Markov) process obtained from X by killing with a positive multiplicative functional (see [23] and Section 5.3 in [2]). We shall call K^q *killing kernel*. If $K^q = U^q_A$, then the multiplicative functional is precisely $(e^{-qt-A_t})_{t\geqslant 0}$.

(ii) If $A_t = \int_0^t c(X_s)\, ds$, $t \geqslant 0$, where $c \in p\mathcal{B}$, then $U^q_A g = U_q(cg)$, $g \in p\mathcal{B}$ and it is known that we have the relation

$$U_{q+\alpha} = W_\alpha + W_\alpha M_c U_{q+\alpha} \quad \text{for all } \alpha > 0,$$

where M_c denotes the operator of multiplication by c.

Let $A' = (A'_t)_{t\geqslant 0}$ be a second continuous additive functional such that $A'_t \leqslant A_t$ for all $t \geqslant 0$ and set

$$K'^q f(x) = U^q_{A'} f(x) = E^x \int_0^\infty e^{-qt} f(X_t)\, dA'_t, \qquad K'^q_1 f(x) = E^x \int_0^\infty e^{-qt-A'_t} f(X_t)\, dA'_t.$$

Proposition 4.5. *Let H be a kernel on $(E,\mathcal{B})$ which is sub-Markovian (i.e., $H1 \leqslant 1$). Let us define the kernels P and P_α, $\alpha > 0$, as*

$$P := K_1'^q H, \qquad P_\alpha := P - \alpha W_\alpha P.$$

If K'^q is a bounded kernel, then the family of kernels $\mathcal{V} = (V_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ defined as

$$V_\alpha = \sum_{k=0}^{\infty} P_\alpha^k W_\alpha, \quad \alpha > 0,$$

is the resolvent of a right (Markov) process with state space E.

Proof. Let $\mathcal{W}' = (W'_\alpha)_{\alpha>0}$ be the resolvent subordinate to $\mathcal{U}$ by the multiplicative functional $(e^{-qt-A'_t})_{t\geqslant 0}$,

$$W'_\alpha f(x) := E^x \int_0^\infty e^{-qt-\alpha A'_t} f(X_t)\,dt, \quad f \in p\mathcal{B}.$$

Note that since $A'_t \leqslant A_t$ it follows that $W_\alpha \leqslant W'_\alpha$ for all $\alpha > 0$ and therefore $\mathcal{E}(\mathcal{W}') \subset \mathcal{E}(\mathcal{W})$. In addition we have $1 - K_1'^q 1 \in \mathcal{E}(\mathcal{W}')$ and $K_1'^q f \in \mathcal{E}(\mathcal{W}')$ for all $f \in bp\mathcal{B}$.

By Proposition 5.2.2 from [2] the family $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a resolvent of kernels on $(E,\mathcal{B})$ and according with Proposition 5.2.3 $\mathcal{V}$ is sub-Markovian if and only if the function $1 - P1$ is $\mathcal{W}$-excessive. From $1 - P1 = (1 - K_1'^q 1) + K_1'^q(1 - H1)$ we get $1 - P1 \in \mathcal{E}(\mathcal{W}') \subset \mathcal{E}(\mathcal{W})$.

The assertion follows now since by Theorem 5.1.6 and Proposition 5.2.4 in [2] $\mathcal{W}$ and respectively $\mathcal{V}$ satisfy condition $(A.1b)$ from Appendix A. $\square$

Corollary 4.6. *With the notations from Proposition 4.5 assume that $K^q f = U_q(cf)$, $K'^q f = U_q(c'f)$, $f \in p\mathcal{B}$, where $c, c' \in bp\mathcal{B}$, $c' \leqslant c$, and H is a kernel on $(E,\mathcal{B})$. Let $\mathcal{L}$ (resp. $\mathcal{L}'$) be the infinitesimal operator of $\mathcal{U}$ (resp. $\mathcal{V}$), regarded as a resolvent of contractions on the space of all bounded $\mathcal{B}$-measurable functions on E. Then $\mathcal{L}$ and $\mathcal{L}'$ have the same domain of definition and*

$$\mathcal{L}' = (\mathcal{L} - q - c) + c'H.$$

Proof. The assertion follows from Lemma 1.4 in [7], because by Remark 4.4(ii) $P_\alpha = W_\alpha M_{c'} H$, hence $V_\alpha = W_\alpha + W_\alpha M_{c'} H V_\alpha$ for all $\alpha > 0$. $\square$

Proposition 4.7. *Let $m \geqslant 2$ and $A = (A_t)_{t\geqslant 0}$ be a continuous additive functional of X such that U_A^q is bounded. Then the following assertions hold.*

(i) *The functional $A^m = (A_t^m)_{t\geqslant 0}$, $A_t^m : \Omega^m \longrightarrow [0,\infty)$, defined as*

$$A_t^m(\omega_1,\dots,\omega_m) = A_t(\omega_1) + \dots + A_t(\omega_m), \quad (\omega_1,\dots,\omega_m) \in \Omega^m,$$

is a continuous additive functional of X^m and its q-potential kernel $U_{A^m}^q$ is bounded and invariant.

(ii) *The projection $\widetilde{K^{q,m}}$ of $U_{A^m}^q$ on $E^{(m)}$, $\widetilde{K^{q,m}} h := \widetilde{U_{A^m}^q (h \circ \pi)}$, $h \in p\mathcal{B}^{(m)}$, is regular excessive with respect to $\mathcal{U}_q^{(m)}$ and it is bounded, $\widetilde{K^{q,m}} 1 \leqslant m U_A^q 1$.*

Proof. It is easy to check that A^m is a continuous additive functional of X^m and for $x = (x_1,\dots,x_m) \in E^m$, $g_1,\dots,g_m \in p\mathcal{B}$ we have

$$U_{A^m}^q(g_1 \otimes \dots \otimes g_m)(x) = E^{x_1} \int_0^\infty e^{-qt} g_1\big(X_t(\omega)\big) P_t g_2(x_2) \cdots P_t g_m(x_m)\, dA_t(\omega) + \cdots$$
$$+ E^{x_m} \int_0^\infty e^{-qt} P_t g_1(x_1) \cdots P_t g_{m-1}(x_{m-1}) g_m\big(X_t(\omega)\big)\, dA_t(\omega).$$

In particular, $U_{A^m}^q 1(x_1,\dots,x_m) \leqslant U_A^q 1(x_1) + \dots + U_A^q 1(x_m) \leqslant m\|U_A^q\|_\infty$. To show that $U_{A^m}^q$ is invariant, we have to prove that if $\pi \in \sigma^m$ and $f = g_1 \otimes \dots \otimes g_m$ then $U_{A^m}^q(f^\#)(\overline{\pi}(x)) = U_{A^m}^q(f^\#)(x)$ for all $x \in E^m$. Since $f^\# = \frac{1}{m!}\sum_{\tau\in\sigma^m} g_{\tau(1)} \otimes \dots \otimes g_{\tau(m)}$ and using the substitution $\tau' = \tau \circ \pi^{-1}$, we get

$$\begin{aligned}
U^q_{A^m}\big(f^\#\big)\big(\overline{\pi}(x)\big) &= \sum_{\tau\in\sigma^m}\frac{1}{m!}\Bigg\{E^{x_{\pi(1)}}\int_0^\infty e^{-qt}g_{\tau(1)}\big(X_t(\omega)\big)P_tg_{\tau(2)}(x_{\pi(2)})\cdots P_tg_{\tau(m)}(x_{\pi(m)})\,dA_t(\omega)+\cdots\\
&\quad + E^{x_{\pi(m)}}\int_0^\infty e^{-qt}P_tg_{\tau(1)}(x_{\pi(1)})\cdots P_tg_{\tau(m-1)}(x_{\pi(m-1)})g_{\tau(m)}\big(X_t(\omega)\big)\,dA_t(\omega)\Bigg\}\\
&= \sum_{\tau'\in\sigma^m}\frac{1}{m!}\Bigg\{E^{x_{\pi(1)}}\int_0^\infty e^{-qt}g_{\tau'(\pi(1))}\big(X_t(\omega)\big)P_tg_{\tau'(\pi(2))}(x_{\pi(2)})\cdots P_tg_{\tau'(\pi(m))}(x_{\pi(m)})\,dA_t(\omega)+\cdots\\
&\quad + E^{x_m}\int_0^\infty e^{-qt}P_tg_{\tau'(\pi(1))}(x_{\pi(1)})\cdots P_tg_{\tau'(\pi(m-1))}(x_{\pi(m-1)})g_{\tau'(\pi(m))}\big(X_t(\omega)\big)\,dA_t(\omega)\Bigg\}\\
&= U^q_{A^m}\big(f^\#\big)(x).
\end{aligned}$$

Assertion (ii) follows by Proposition 4.1 because the kernel $U^q_{A^m}$ is regular excessive with respect to $\mathcal{U}^m_q$, being the q-potential kernel of a continuous additive functional of X^m. $\square$

4.3. Markov processes on S

A resolvent of kernels $\mathcal{V}=(V_\alpha)_{\alpha>0}$ on $(S,\mathcal{B}(S))$ is called *diagonal* if V_α is a diagonal kernel for every $\alpha>0$.

Let $\widetilde{\mathcal{U}}=(\widetilde{U}_\alpha)_{\alpha>0}$ be the diagonal resolvent on $(S,\mathcal{B}(S))$ defined as $\widetilde{U}_\alpha:=\bigoplus_{m\geqslant1}U^{(m)}_\alpha$, i.e., for all $\alpha>0$, $g\in p\mathcal{B}(S)$, and $m\geqslant1$

$$\widetilde{U}_\alpha g|_{E^{(m)}} := U^{(m)}_\alpha(g|_{E^{(m)}}).$$

According with [15] and [24], a semigroup of sub-Markovian kernels $(T_t)_{t\geqslant0}$ on $(S,\mathcal{B}(S))$ is called *branching semigroup* provided that T_t is a branching kernel for all $t>0$. A Markov process whose transition function is a branching semigroup is called *branching process*.

Remark 4.8.

(i) Recall from [24] the following description of a discrete branching process: *An initial particle starts at a point of E and moves according to the base process X until a random time* (*defined by killing X as described in assertion* (i) *of Remark* 4.4) *when it splits into a random number m of new particles, its direct descendants, placed in E. Each direct descendant starts at the terminal position of the parent particle and moves on according to the m independent copies of X and so on.*

(ii) The family $\widetilde{\mathcal{U}}$ is the resolvent associated with the semigroup of kernels $(\widetilde{P}_t)_{t\geqslant0}$, where $\widetilde{P}_tg|_{E^{(m)}}:=P^{(m)}_t(g|_{E^{(m)}})$, $g\in p\mathcal{B}(S)$. The family $(\widetilde{P}_t)_{t\geqslant0}$ is a branching semigroup. Indeed, if $\varphi\in p\mathcal{B}$, $\varphi\leqslant1$, then $\widehat{\widetilde{P}_t\widehat{\varphi}|_E}=\widehat{P_t\varphi}=\widetilde{P}_t\widehat{\varphi}$.

As a consequence of Proposition 4.2 we can state the first existence result of a Markov process with state space S, the space of finite configurations of E.

Corollary 4.9. *The family $\widetilde{\mathcal{U}}=(\widetilde{U}_\alpha)_{\alpha>0}$ is the resolvent of a right* (*Markov*) *process with state space S.*

Proof. Since by Lemma 2.3 S is a Lusin topological space, we have to show that $\widetilde{\mathcal{U}}$ verifies condition $(A.1b)$ from Appendix A. This follows easily because the restriction of $\widetilde{\mathcal{U}}$ to each $E^{(m)}$ satisfies $(A.1b)$, being the resolvent of a right process with state space $E^{(m)}$ (according with Proposition 4.2). $\square$

Let $A=(A_t)_{t\geqslant0}$ be a continuous additive functional of X and $q>0$ be such that U^q_A is a bounded kernel on $(E,\mathcal{B})$. By Proposition 4.7 applied for each m with $A^m=(A^m_t)_{t\geqslant0}$ and $A'^m:=(\frac{1}{m}A^m_t)_{t\geqslant0}$, we obtain two kernels $\widetilde{K^{q,m}}$ and $\widetilde{K'^{q,m}}$ respectively.

Let $(\widetilde{K^{q,m}_\alpha})_{\alpha>0}$ (resp. $(\widetilde{K'^{q,m}_\alpha})_{\alpha>0}$) be the sub-Markovian resolvent on $(E^{(m)},\mathcal{B}^{(m)})$ having $\widetilde{K^{q,m}}$ (resp. $\widetilde{K'^{q,m}}$) as initial kernel. The existence of these resolvents is ensured by Corollary 1.1.13 from [2], since $\widetilde{K'^{q,m}}1\leqslant\widetilde{K^{q,m}}1<\infty$ and assertion (ii) of Proposition 4.7 implies that $\widetilde{K^{q,m}}$ and $\widetilde{K'^{q,m}}$ satisfy the complete maximum principle.

We can state now the existence result for the measure-valued branching type process generated by the base process X, the killing kernel K^q, and the branching kernel $\widehat{B}$.

Theorem 4.10. *Let $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ be a sub-Markovian kernel, $\widehat{B}$ the branching kernel on $(S, \mathcal{B}(S))$ given by Proposition 3.2, and put*

$$\widetilde{K_1^q} := \bigoplus_{m\geqslant 1} \widetilde{K_1^{q,m}}, \qquad \widetilde{K_1'^q} := \bigoplus_{m\geqslant 1} \widetilde{K_1'^{q,m}}, \qquad \widetilde{P} := \widetilde{K_1'^q}\widehat{B}.$$

Then the following assertions hold.

(i) *$\widetilde{K_1^q}$ and $\widetilde{K_1'^q}$ are subordination operators with respect to $\widetilde{\mathcal{U}}_q$.*

(ii) *If $\widetilde{\mathcal{W}} = (\widetilde{W}_\alpha)_{\alpha>0}$ (resp. $\widetilde{\mathcal{W}}' = (\widetilde{W}'_\alpha)_{\alpha>0}$) is the subordinate resolvent associated with $\widetilde{K_1^q}$ (resp. $\widetilde{K_1'^q}$), then the family of kernels $\widetilde{\mathcal{V}} = (\widetilde{V}_\alpha)_{\alpha>0}$ (resp. $\widetilde{\mathcal{V}}' = (\widetilde{V}'_\alpha)_{\alpha>0}$) defined as*

$$\widetilde{V}_\alpha := \sum_{k=0}^{\infty} (\widetilde{P}_\alpha)^k \widetilde{W}_\alpha \quad \left(\text{resp. } \widetilde{V}'_\alpha := \sum_{k=0}^{\infty} \big(\widetilde{P'}_\alpha\big)^k \widetilde{W}'_\alpha\right), \quad \alpha > 0,$$

is the resolvent of a right (Markov) process with state space S, where $\widetilde{P}_\alpha := \widetilde{P} - \alpha \widetilde{W}_\alpha \widetilde{P}$ (resp. $\widetilde{P}'_\alpha := \widetilde{P} - \alpha \widetilde{W}'_\alpha \widetilde{P}$).

Proof. The first assertion holds since the family $(\widetilde{K_\alpha^q})_{\alpha>0}$ (resp. $(\widetilde{K_\alpha'^q})_{\alpha>0}$) defined as

$$\widetilde{K_\alpha^q} := \bigoplus_{m\geqslant 1} \widetilde{K_\alpha^{q,m}} \quad \left(\text{resp. } \widetilde{K_\alpha'^q} := \bigoplus_{m\geqslant 1} \widetilde{K_\alpha'^{q,m}}\right)$$

is a sub-Markovian resolvent having $\widetilde{K^q} = \bigoplus_{m\geqslant 1} \widetilde{K^{q,m}}$ (resp. $\widetilde{K'^q} = \bigoplus_{m\geqslant 1} \widetilde{K'^{q,m}}$) as initial kernel and by Proposition 4.7 we get that $\widetilde{K^q}$ (resp. $\widetilde{K'^q}$) is a regular excessive kernel with respect to $\widetilde{\mathcal{U}}_q$ and $\widetilde{K'^q}1 \leqslant \widetilde{K^q}1 < \infty$. Assertion (ii) follows now applying Proposition 4.5 for the resolvent $\widetilde{\mathcal{U}}$ on $(S, \mathcal{B}(S))$ and taking $H := \widehat{B}$. $\square$

Proposition 4.11. *Let $\widetilde{X}$ be the right Markov process with state space S, having the resolvent family $\widetilde{\mathcal{U}} = (\widetilde{U}_\alpha)_{\alpha>0}$ (the process given by Corollary 4.9). Assume that the base process X is transient (or equivalently, the resolvent $\mathcal{U}$ is proper, i.e., there exists $f \in p\mathcal{B}$, $f > 0$, such that $Uf := \sup_{\alpha>0} U_\alpha f$ is a bounded function). Then all the above results hold also for $q = 0$, where $\mathcal{U}_0 = \mathcal{U}$. The transition function $(\widetilde{T}_t)_{t\geqslant 0}$ of the process on S, obtained by killing $\widetilde{X}$ with $\widetilde{K_1^0}$, is a branching semigroup. The restriction to E of $(\widetilde{T}_t)_{t\geqslant 0}$ is the transition function of the process obtained by killing X with K_1^0.*

Proof. If $\varphi \in p\mathcal{B}$, $\varphi \leqslant 1$ and $x = (x_1, \ldots, x_m) \in E^m$ then $\widetilde{T}_t\widehat{\varphi}(\widetilde{x}) = E^x[e^{-A_t^m}\widehat{\varphi}(X_t^m)] = \prod_{i=1}^m E^{x_i}[e^{-A_t}\varphi(X_t)]$. Therefore $\widetilde{T}_t$ maps multiplicative functions into multiplicative functions, hence it is a branching kernel according with Remark 3.1. In particular, if $m = 1$, then $\widetilde{T}_t\varphi(x) = E^x[e^{-A_t}\varphi(X_t)]$, $x \in E$. $\square$

Remark 4.12.

(i) Proposition 4.11 gives an example of transformation of a branching Markov process, as it was developed in [16, Chapter V]. More precisely, the transformation of $\widetilde{X}$ is made here by killing a branching process with a multiplicative functional of "branching type" (in the sense of Definition 5.1 from [16]) and the above proof of the branching property of $\widetilde{T}_t$ is similar with the proof of the implication (ii) $\Longrightarrow$ (i) of Theorem 5.1 from [16, page 148].

(ii) Taking into account the above discussion, we call the Markov processes obtained in assertion (ii) of Theorem 4.10 *discrete branching type processes* with base process X, associated with the killing kernels K^q and K'^q, and the branching kernel $\widehat{B}$.

(iii) By assertion (ii) of Theorem 4.10, the kernel $\widetilde{P}$ is a subordination operator with respect to $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{V}}'$. The associated subordinate resolvent is clearly $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}'$ respectively. For other related examples of subordination operators on a direct sum see [6].

We consider now a particular case as in Corollary 4.6, which allows to express the relations between the infinitesimal operators. Recall that $\mathcal{L}$ denotes the generator of the given base process X with state space E, i.e., it is the infinitesimal operator of the resolvent $\mathcal{U}$ on the space of all bounded $\mathcal{B}(S)$-measurable functions on S. Let $\widetilde{\mathcal{V}}$ be the resolvent given by Theorem 4.10 with

$$A_t^m(\omega_1, \ldots, \omega_m) := \frac{1}{m} \sum_{i=1}^m \int_0^t c\big(X_s(\omega_i)\big)\, ds \tag{4.3}$$

for all $t \geqslant 0$ and $m \geqslant 1$, where $c \in bp\mathcal{B}$.

Corollary 4.13. *Let $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{L}}'$) be the infinitesimal operator of $\widetilde{\mathcal{U}}$ (resp. $\widetilde{\mathcal{V}}$), regarded as a resolvent of contractions on the space of all bounded $\mathcal{B}(S)$-measurable functions on S. Then the following assertions hold.*

(i) *The operators $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$ have the same domain of definition and*

$$\widetilde{\mathcal{L}}' = (\widetilde{\mathcal{L}} - q - \bar{c}) + \bar{c}\widehat{B},$$

where $\bar{c} \in bp\mathcal{B}(S)$ is such that for all m $\bar{c}|_{E^{(m)}} = \widetilde{c_m}$, with $c_m(x_1, \dots, x_m) := \frac{1}{m}(c(x_1) + \dots + c(x_m))$.

(ii) *Let u be a $\mathcal{B}$-measurable function, $|u| \leqslant 1$. Then u belongs to the domain of $\mathcal{L}$ if and only if $\widehat{u}$ lies in the domain of $\widetilde{\mathcal{L}}$. In this case we have*

$$\widetilde{\mathcal{L}}'\widehat{u}|_E = (\mathcal{L} - q - c)u + cB\widehat{u}.$$

In particular, if B is given by (3.2), *then*

$$\widetilde{\mathcal{L}}'\widehat{u}|_E = (\mathcal{L} - q - c)u + c\sum_{k \geqslant 1} q_k u^k.$$

Proof. Assertion (i) is a direct consequence of Theorem 4.10 and Corollary 4.6, observing that in this case the continuous additive functional A^m of X^m in Proposition 4.7 is given by (4.3).

(ii) The fact that $\widehat{u}$ belongs to the domain of $\widetilde{\mathcal{L}}$, provided that u is in the domain of $\mathcal{L}$, holds by induction from the following claim which may be easily checked (see [25, page 52] and [21] for further developments):

Let $(E_i, \mathcal{B}_i)$, $i = 1, 2$, be two measurable spaces. For each i let further $(P_t^i)_{t \geqslant 0}$ be a measurable semigroup of kernels on $(E_i, \mathcal{B}_i)$ with bounded potential kernel U^i, $U^i f := \int_0^\infty P_t^i f\,dt$, $f \in p\mathcal{B}_i$. If v_i is the potential of a function $f_i \in bp\mathcal{B}_i$, i.e., $v_i = U^i f_i$, $i = 1, 2$, then $v_1 \otimes v_2$ is the potential of $v_1 \otimes f_2 + f_1 \otimes v_2$ with respect to the potential kernel of the product semigroup $(P_t^1 \times P_t^2)_{t \geqslant 0}$ on $(E_1 \times E_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$.

Conversely, if $\widehat{u}$ belongs to the domain of $\widetilde{\mathcal{L}}$, $\widehat{u} = \widetilde{U}_\beta g$ for some $\beta > 0$ and $g \in b\mathcal{B}(S)$, then clearly $u = U_\beta(g|_E)$.

The first equality from (ii) follows now from (i), since by Proposition 3.2 we have $\widehat{B}\widehat{u}|_E = B\widehat{u}$ and because $\widetilde{\mathcal{L}}\widehat{u}|_E = \mathcal{L}u$. □

Remark 4.14. A perturbation method with branching kernels of the type described in Theorem 4.10 and Corollary 4.13 was established by K. Janssen in the frame of the balayage spaces (on a metrizable locally compact space); cf. [17]. Note that the corresponding perturbed operator is there $\widetilde{\mathcal{L}} + \widehat{B}$ and therefore the associated resolvent is no longer necessarily sub-Markovian.

Corollary 4.15. *Let u be a $\mathcal{B}$-measurable function, $|u| \leqslant 1$.*

(i) *If $\widehat{u}$ is an $\widetilde{\mathcal{L}}'$-harmonic function, i.e., $\widetilde{\mathcal{L}}'\widehat{u} = 0$, then*

$$(\mathcal{L} - q - c)u + cB\widehat{u} = 0. \tag{4.4}$$

(ii) *If $q_k \in p\mathcal{B}$, $k \geqslant 1$, with $\sum_{k \geqslant 1} q_k \leqslant 1$, then u is a solution of the nonlinear equation*

$$(\mathcal{L} - q - c)u + c\sum_{k \geqslant 1} q_k u^k = 0, \tag{4.5}$$

provided that $\widehat{u}$ is an $\widetilde{\mathcal{L}}'$-harmonic function, with B given by (3.2).

Remark 4.16.

(i) Let Λ_o be the nonlinear operator given by the left-hand side of the equality (4.4), i.e., $\Lambda_o u := (\mathcal{L} - q - c)u + cB\widehat{u}$. Assume that $\widehat{u}$ is an invariant function for $\widetilde{\mathcal{V}}$. Then clearly $\widehat{u}$ is $\mathcal{L}'$-harmonic and thus, by assertion (i) of Corollary 4.15, the function u is a solution of the nonlinear equation $\Lambda_o u = 0$. Therefore Corollary 4.15 is an extension of Proposition 1 from [20]. Recall that in [20] the above result was used to show that solutions (not necessarily unique) of the Dirichlet problem associated with the nonlinear equation given by (4.5) (on a regular open set with respect to a path continuous strong Feller process) may be obtained in terms of an appropriate branching semigroup like the transition function of the right Markov processes given by Theorem 4.10. To keep the size of this paper within reasonable limits, we shall only briefly present this approach related to our results in the Addendum below; we thank the referee for suggesting us to make this completion.

(ii) The results obtained in Theorem 4.10, Corollary 4.13, and Corollary 4.15 have several interesting extensions we shall discuss in a forthcoming paper. One can consider killing kernels K^q which are only regular strongly supermedian instead of being excessive. In this case the continuous additive functionals will be replaced by the positive left continuous ones; for details see [14] and [2]. Several technical problems will occur since it is necessary to enlarge the measurability from the Borel σ-algebra to the "nearly Borel" one. Note finally that it is possible to consider the infinitesimal operators acting on L^p spaces instead of the bounded functions and to exploit the Kato class results established in [3].

5. Application: continuous and discrete branching

Let Y be a right (Markov) process with state space a Lusin topological space F, called *spatial motion*. We fix a *branching mechanism*, that is, a function $\Phi : F \times [0,\infty) \longrightarrow \mathbb{R}$ of the form

$$\Phi(x,\lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty \bigl(1 - e^{-\lambda s} - \lambda s\bigr) N(x, ds)$$

where $c \geqslant 0$ and b are bounded $\mathcal{B}$-measurable functions and $N : p\mathcal{B}((0,\infty)) \longrightarrow p\mathcal{B}(F)$ is a kernel such that $N(u \wedge u^2) \in bp\mathcal{B}$. Examples of branching mechanisms are $\Phi(\lambda) = -\lambda^\alpha$ for $1 < \alpha \leqslant 2$ and $\Phi(\lambda) = \lambda^\alpha$ for $0 < \alpha < 1$.

We present now the measure-valued branching Markov process associated with the spatial motion Y and the branching mechanism Φ, the (Y,Φ)-superprocess, a Borel right process with state space $M(F)$, the space of all positive finite measures on $(F, \mathcal{B}(F))$, endowed with the weak topology (cf. [13] and [11], see also [1]).

For each $f \in bp\mathcal{B}$ the equation

$$v_t(x) = T_t f(x) + \int_0^t T_s\bigl(x, \Phi(\cdot, v_{t-s})\bigr)\, ds, \quad t \geqslant 0,\ x \in F,$$

has a unique solution $(t,x) \longmapsto N_t f(x)$ jointly measurable in (t,x) such that $\sup_{0 \leqslant s \leqslant t} \|v_s\|_\infty < \infty$ for all $t > 0$; we have denoted by $(T_t)_{t \geqslant 0}$ the transition function of the spatial motion Y. The mappings $f \longmapsto N_t f$ form a nonlinear semigroup of operators on $bp\mathcal{B}(F)$. The above equation is formally equivalent with

$$\begin{cases} \dfrac{d}{dt} v_t(x) = \mathsf{L} v_t(x) + \Phi\bigl(x, v_t(x)\bigr), \\ v_0 = f, \end{cases}$$

where L is the infinitesimal generator of the spatial motion Y. For every $t \geqslant 0$ there exists a unique kernel P_t on $(M(F), \mathcal{M}(F))$ such that

$$P_t(e_f) = e_{N_t f}, \quad f \in bp\mathcal{B}(F). \tag{5.1}$$

Since the family $(N_t)_{t \geqslant 0}$ is a (nonlinear) semigroup on $bp\mathcal{B}(F)$, $(P_t)_{t \geqslant 0}$ is a linear semigroup of kernels on $M(F)$. It turns out that (see, e.g., [13] and [1]) $(P_t)_{t \geqslant 0}$ is the transition function of a right (Markov) process with state space $M(F)$, called *(Y,Φ)-superprocess*.

Remark 5.1.

(i) The transition function $(P_t)_{t \geqslant 0}$ of the (Y,Φ)-superprocess is a branching semigroup on $M(F)$, i.e., P_t is branching kernel on $M(F)$ for all $t > 0$. *Indeed, the assertion follows by* (5.1), *using* (2.1).

(ii) The following suggestive description of a superprocess was stated in [12, page 55]: "A measure-valued Markov process describes the evolution of a random cloud. The branching property means that any parts of the cloud at time t do not interact after t."

We can apply now the results from Section 4, starting with the (Y,Φ)-superprocess as base process with state space $E := M(F)$.

Corollary 5.2. *Let $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ be a sub-Markovian kernel and K^q a killing kernel on E. Then there exists a discrete branching type process with base process the (Y,Φ)-superprocess, associated to the killing kernel K^q and the branching kernel $\widehat{B}$, and having as state space the set of finite configurations of positive finite measures on F.*

Final remark. By Remark 5.1 and taking into account the interpretation of a branching process given in assertion (i) of Remark 4.8, one can think that the process obtained in Corollary 5.2 describes the evolution of a random cloud controlled not only by a branching mechanism Φ but also by a discrete branching, in a new dimension, along which the splitting into a random number of clouds takes place, commanded by a killing kernel K^q and a branching kernel $\widehat{B}$.

6. Addendum: The semigroup approach and the nonlinear Dirichlet problem

6.1. The construction of a branching semigroup on S

We assume further, as in Proposition 4.11, that the base process X is transient.

Proposition 6.1. *Suppose that $c \in bp\mathcal{B}$ and $B : bp\mathcal{B}(S) \longrightarrow bp\mathcal{B}$ is a sub-Markovian kernel such that*

$$\sup_{x\in E} Bl_1(x) < \infty. \tag{6.1}$$

Then there exists a branching semigroup $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ on S, having the following property: If $u \in p\mathcal{B}$, $u \leqslant 1$, is such that $\widehat{u}$ is an invariant function with respect to this semigroup, then u belongs to the domain of $\mathcal{L}$ (the infinitesimal generator of the base process X) and

$$(\mathcal{L} - c)u + cB\widehat{u} = 0.$$

In particular, if B is given by (3.2), *then u is the solution of the nonlinear equation*

$$(\mathcal{L} - c)u + c\sum_{k\geqslant 1} q_k u^k = 0.$$

Note that (6.1) is precisely Condition 4.1.2 from [24] and if B is given by (3.2) then (6.1) is equivalent with

$$\sup_{x\in E} \sum_{k\geqslant 1} kq_k(x) < \infty.$$

We follow the approach of [24], solving an appropriate integral equation. Denote by $\mathcal{B}_1$ the set of all functions $\varphi \in p\mathcal{B}$ such that $\varphi \leqslant 1$. Recall that a map $H : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ is called *absolutely monotonic* provided that there exists a sub-Markovian kernel $\mathbf{H} : bp\mathcal{B}(S) \longrightarrow bp\mathcal{B}$ such that $H\varphi = \mathbf{H}\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_1$. By Proposition 3.2 we have:

$(AD1)$ *A map $H : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ is absolutely monotonic if and only if there exists a branching kernel $\widehat{\mathbf{H}}$ on S such that $\widehat{\mathbf{H}}\widehat{\varphi} = \widehat{H\varphi}$ for all $\varphi \in \mathcal{B}_1$.*

We also have (cf. Lemma 2.2 and Theorem 1 from [24]):

$(AD2)$ *If H, K are absolutely monotonic then their composition HK is also absolutely monotonic and $\widehat{\mathbf{HK}} = \widehat{\mathbf{H}}\widehat{\mathbf{K}}$. The map $H \longmapsto \widehat{\mathbf{H}}$ is a bijection between the set of all absolutely monotonic mappings and the set of all branching kernels on S.*

Let $(T_t)_{t\geqslant 0}$ be the transition function of the process having $\mathcal{L} - c$ as infinitesimal generator, it is expressed using the Feynman–Kac formula:

$$T_t f(x) = E^x\big[e^{-\int_0^t c(X_s)\,ds} f(X_t)\big], \quad f \in bp\mathcal{B}. \tag{6.2}$$

Proposition 6.2. *Assume that* (6.1) *holds. For every $\varphi \in \mathcal{B}_1$ the equation*

$$w_t(x) = T_t\varphi(x) + \int_0^t T_s(cB\widehat{w}_{t-s})(x)\,ds, \quad t \geqslant 0,\ x \in E, \tag{6.3}$$

has a unique solution $(t,x) \longmapsto H_t\varphi(x)$ jointly measurable in (t,x), such that $H_t\varphi \in \mathcal{B}_1$ and the following assertions hold.

(i) *The mapping $\varphi \longmapsto H_t\varphi$ is absolutely monotonic for all $t \geqslant 0$.*
(ii) *The family $(H_t)_{t\geqslant 0}$ is a nonlinear semigroup of operators on $\mathcal{B}_1$.*

Sketch of the proof of Proposition 6.2. As in [24, inequality (4.11)], one can see that if $\varphi, \psi \in \mathcal{B}_1$ and $\mu \in S$ then

$$\big|\widehat{\varphi}(\mu) - \widehat{\psi}(\mu)\big| \leqslant l_1(\mu)\|\varphi - \psi\|_\infty.$$

If $K\varphi := cB\widehat{\varphi}$, then from (6.1) and the above inequality we conclude that the mapping $\varphi \longmapsto K\varphi$ is Lipschitz. The uniqueness for Eq. (6.3) follows now by Gronwall's lemma, while the semigroup property (assertion (ii)) is its consequence.

Define inductively the operators H_t^n (see [24, page 250]) as

$$H_t^0\varphi = T_t\varphi, \qquad H_t^{n+1}\varphi = T_t\varphi + \int_0^t T_s K H_{t-s}^n \varphi\, ds, \quad \varphi \in \mathcal{B}_1.$$

It turns out that the sequence $(H_t^n\varphi)_n$ is increasing and H_t^n is absolutely monotonic for all t and n. Note that the proof of $H_t^n 1 \leqslant 1$ uses the above mentioned Feynman–Kac expression (6.2) of T_t. For $x \in E$ and $\varphi \in \mathcal{B}_1$ we set

$$H_t\varphi(x) = \sup_n H_t^n\varphi(x).$$

One can check that the sequence of kernels $(\mathbf{H}_t^n)_n$ (where $\mathbf{H}_t^n$ is such that $H_t^n\varphi = \mathbf{H}_t^n\widehat{\varphi}$) is increasing and as a consequence H_t is absolutely monotonic for all t. We conclude that $(t,x) \longmapsto H_t\varphi(x)$ is the claimed solution of (6.3). $\square$

Proof of Proposition 6.1. For each $t > 0$ let H_t be the absolutely monotonic operator given by Proposition 6.2. Assertion $(AD1)$ implies the existence of a branching kernel $\widehat{\mathbf{H}}_t$ on $(S, \mathcal{B}(S))$, $t \geqslant 0$, such that $H_t\varphi = \widehat{\mathbf{H}_t\widehat{\varphi}}|_E$ for all $\varphi \in \mathcal{B}_1$. By assertion (ii) of Proposition 6.2 and using $(AD2)$ it follows that the family $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ is a semigroup of branching kernels on S.

If $u \in p\mathcal{B}$, $u \leqslant 1$, is such that $\widehat{u}$ is an invariant function with respect to this semigroup, then clearly $H_t u = u$ for all $t > 0$ and therefore, using (6.3), $u = T_t u + \int_0^t T_s(cB\widehat{u})\, ds$. Letting $t \to \infty$, it follows that $u = v + W(cB\widehat{u})$, where $v := \inf_{t>0} T_t u = \lim_{t\to\infty} T_t u$. In particular, $\alpha W_\alpha v = v$, hence v and u belong to the domain of $\mathcal{L} - c$ which coincides with the domain of $\mathcal{L}$, according with Corollary 4.6. Let $f := cB\widehat{u}$ and $w := Wf$. Then $u = v + w = W_\alpha(\alpha u + f)$, $(\mathcal{L} - c)u = \alpha W_\alpha(\alpha u + f) - (\alpha u + f) = -cB\widehat{u}$. $\square$

Remark 6.3. Let $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ be the branching semigroup on S given by Proposition 6.1 and H_t the absolutely monotonic operator such that $H_t\varphi = \mathbf{H}_t\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_1$. If $f \in bp\mathcal{B}$ define the function $V_t f \in bp\mathcal{B}$ as $V_t f := -\ln H_t(\exp(-f))$. It is easy to see that the family $(V_t)_{t\geqslant 0}$ is a (nonlinear) semigroup on $bp\mathcal{B}$ and the following equality similar to (5.1) holds

$$\mathbf{H}_t(e_f) = e_{V_t f}, \quad f \in bp\mathcal{B}.$$

Consequently, the so-called semigroup approach for the continuous branching (due to E.B. Dynkin and P.J. Fitzsimmons and briefly presented in Section 5) is analogue to the above construction of the branching semigroup (developed by N. Ikeda, M. Nagasawa, S. Watanabe, and M.L. Silverstein) in the discrete branching case.

6.2. The nonlinear Dirichlet problem

Assume that Y is a transient, path continuous right Markov process with state space a Lusin topological space F. Let D be an open subset of F and suppose that every point of the boundary ∂D of D is *regular*, i.e. $P^y(T_{F\setminus D} = 0) = 1$ for all $y \in \partial D$. Following [10, Chapter X, Section 2], we may consider the stopped process at the boundary of D: $\overline{Y}_t := Y_{t\wedge T}$, where T is the entry time of ∂D, $T := \inf\{t \geqslant 0\colon\ Y_t \in \partial D\}$ and assume that $P^x(T < \zeta) = 1$ for all $x \in \overline{D}$. Let $c \in bp\mathcal{B}(D)$ and extend it to F with zero on $F \setminus D$.

We set $E := \overline{D}$ and for $\beta > 0$ take as base process X on E the β-level subprocess of $\overline{Y}_t$. Let $(T_t)_{t\geqslant 0}$ be the semigroup given by (6.2). Since c equals zero on ∂D we get

$$\exists \lim_{t\to\infty} T_t f(x) = E^x\bigl[e^{-\int_0^T c(Y_s)\, ds} f(Y_T)\bigr] =: P_T^c f(x), \quad x \in E,\ f \in bp\mathcal{B}(F). \tag{6.4}$$

We also assume that if $\varphi \in bpC(\partial D)$ then

$$\lim_{D\ni x\to y} P_T^c\varphi(x) = \varphi(y) \quad \text{for all y} \in \partial D; \tag{6.5}$$

see Theorem 13.1 from [10] for situations when this property holds.

We need some regularity assumptions. Suppose that the resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ of Y is strong Feller (i.e., $W_\alpha f \in C(F)$ for all $f \in bp\mathcal{B}(F)$ and $\alpha > 0$) and the following *domination property* holds: if $f \in bp\mathcal{B}(F)$ and $v \in \mathcal{E}(\mathcal{W}_\beta)$, $v \leqslant W_\beta f$, then v is continuous.

The next result is a version of Proposition 3 from [20]. Recall that $\mathcal{L}$ is the infinitesimal generator of X.

Proposition 6.4. *Let $B : bp\mathcal{B}(S) \longrightarrow bp\mathcal{B}$ be a sub-Markovian kernel satisfying* (6.1) *and $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ the branching semigroup on S given by Proposition* 6.1. *If $\varphi \in bpC(\partial D)$, $\varphi \leqslant 1$, is such that*

$$\exists \lim_{t\to\infty} H_t\varphi(x) =: u(x), \quad x \in E, \tag{6.6}$$

then u is a solution of the nonlinear Dirichlet problem

$$\begin{cases} (\mathcal{L} - c)u + cB\widehat{u} = 0 & \textit{on } D, \\ \lim\limits_{D\ni x\to y} u(x) = \varphi(y) & \textit{for all } y \in \partial D. \end{cases}$$

Proof. From (6.6) it follows that the function $\widehat{u}$ is invariant with respect to $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$, while by Proposition 6.1 we deduce that u belongs to the domain of $\mathcal{L}$ and $(\mathcal{L}-c)u + cB\widehat{u} = 0$. From $H_t\varphi = T_t\varphi + \int_0^t T_s cB\widehat{H_{t-s}\varphi}\,ds$, letting $t\to\infty$, and using (6.4) and (6.6), we get $u = P_T^c\varphi + v$, where $v := \lim_{t\to\infty} T_s cB\widehat{H_{t-s}\varphi}\,ds$. Since c equals zero on ∂D we have $v \leqslant \int_0^\infty T_s c\,ds \leqslant E^{\cdot}\int_0^T e^{-\beta s}c(Y_s)\,ds = W_\beta c - P_T^\beta W_\beta c$. By the regularity assumptions it follows that $W_\beta c$, $P_T^\beta W_\beta c \in bC(F)$. Since $W_\beta c = P_T^\beta W_\beta c$ on $F\setminus D$, we deduce $\lim_{D\ni x\to y} v(x) = 0$ for all $y\in\partial D$ and from (6.5) we conclude that $\lim_{D\ni x\to y} u(x) = \lim_{D\ni x\to y} P_T^c\varphi(x) = \varphi(y)$. $\square$

Remark. Under certain additional assumptions, the existence of the limit (6.6) is proved in [20], in the case when B is given by (3.2).

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Appendix A

A.1. Complements on the sub-Markovian resolvents of kernels

We present a result on the existence of a right process having a given sub-Markovian resolvent of kernels (see Corollary 1.8.12 in [2] and also [5] for the nontransient case).

The following assertions are equivalent for a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on a Lusin topological space E.

$(A.1a)$ $\mathcal{U}$ is the resolvent of a right process with state space E.
$(A.1b)$ For one (and therefore for all) $\beta > 0$ we have:
- The convex cone $\mathcal{E}(\mathcal{U}_\beta)$ is stable for the pointwise infimum, generates the Borel σ-algebra $\mathcal{B}$ and $1\in\mathcal{E}(\mathcal{U}_\beta)$;
- If $\mu\circ U_\beta$ and ξ are two $\mathcal{U}_\beta$-excessive measures (the first one being the potential of the measure μ) and $\xi\leqslant\mu\circ U_\beta$ then ξ is also the potential of a measure;
- Every open subset of E is finely open with respect to $\mathcal{E}(\mathcal{U}_\beta)$; recall that the *fine topology* is the topology on E generated by all $\mathcal{U}_\beta$-excessive functions.

We assume further that $\mathcal{U}$ is the resolvent of a right (Markov) process X with state space E.

A *continuous additive functional* of the process X is a positive, increasing, continuous process $A = (A_t)_{t\geqslant 0}$ such that $A_t\in p\mathcal{F}_t$, $A_t<\infty$ on $[0,\zeta)$, $A_t = A_\zeta$ if $t\geqslant\zeta$ and $A_{t+s} = A_t + A_s(\theta_t)$ for all $s,t\geqslant 0$; for details see, e.g., [23] and also [8]. If $q\geqslant 0$ then the *q-potential kernel* of $A = (A_t)_{t\geqslant 0}$ is defined as

$$U_A^q f(x) = E^x\int_0^\infty e^{-qt} f(X_t)\,dA_t,\quad f\in p\mathcal{B},\ x\in E,$$

and it is a $\mathcal{U}_q$-excessive kernel provided it is proper.

Subordination operators and subordinate resolvents (cf. Chapter 5 in [2]). Assume that $\mathcal{U}$ is proper. A kernel P on $(E,\mathcal{B}^u)$ is called *subordination operator* (with respect to $\mathcal{U}$) provided that $Pu\leqslant u$ and the function $\inf(u, Pu + v - Pv + Pf)$ is $\mathcal{U}$-excessive for all $u, v\in\mathcal{E}(\mathcal{U})$ with $v<\infty$ and $f\in p\mathcal{B}$. If P is a subordination operator with respect to $\mathcal{U}$ we denote by E_P the set defined by

$$E_P = \{x\in E \mid \text{there exists } s\in\mathcal{E}(\mathcal{U}) \text{ with } Ps(x) < s(x)\}.$$

A second sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E,\mathcal{B}^u)$ is called *exactly subordinate* to $\mathcal{U}$ provided that $U'_\alpha\leqslant U_\alpha$ for all $\alpha>0$ and $Uf - U'f$ is $\mathcal{U}$-excessive for every $f\in p\mathcal{B}^u$ with $Uf<\infty$.

Let P be a subordination operator with respect to $\mathcal{U}$. Then there exists a sub-Markovian resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ on $(E,\mathcal{B}^u)$ such that $\mathcal{W}$ is exactly subordinate to $\mathcal{U}$ and $Wf = Uf - PUf$ for all $f\in p\mathcal{B}^u$ with $Uf<\infty$. The sub-Markovian resolvent $\mathcal{W}$ is called *generated* by P.

A.2. Proofs of the results from Section 2

Proof of Lemma 2.2. By assertion (ii) of Remark 2.1 it is sufficient to prove the claimed equality and using a monotone class argument we may assume that $f = g_1\otimes\cdots\otimes g_m$ with $g_i\in p\mathcal{B}$, $i\in\{1,\dots,m\}$. Since $f^\# = \frac{1}{m!}\sum_{\tau\in\sigma^m} g_{\tau(1)}\otimes\cdots\otimes g_{\tau(m)}$ we have

$$N^m\big(f^\#\big) = \frac{1}{m!}\sum_{\tau\in\sigma^m} N^m(g_{\tau(1)}\otimes\cdots\otimes g_{\tau(m)}) = \frac{1}{m!}\sum_{\tau\in\sigma^m} N(g_{\tau(1)})\otimes\cdots\otimes N(g_{\tau(m)})$$
$$= (Ng_1\otimes\cdots\otimes Ng_m)^\# = \big(N^m f\big)^\#. \qquad\square$$

We denote by π the canonical projection from E^m to $E^{(m)}$.

Remark A.2.1. The following assertions hold.

(i) Let $M\subset E^m$. If $M\in\mathcal{B}^m$ (resp. M is open) then $M^\#\in\mathcal{B}^m$ (resp. $M^\#$ is open). If $M=M^\#$ then $\pi(M^c)=\pi(M)^c$.
(ii) (Cf. [9, page 52].) The canonical projection π is an open mapping.

Indeed, we notice first that if $\tau\in\sigma^m$ then $\bar\tau$ is a topological homemorphism of E^m. Consequently, if $M\in\mathcal{B}^m$ (resp. M is an open subset of E^m) then $\bar\tau(M)$ has the same property. The first assertion of (i) follows now since $M^\# = \bigcup_{\tau\in\sigma^m}\bar\tau(M)$, while the second one is a straightforward verification. If $D\in\mathcal{T}^m$ then $\pi(D)=\pi(D^\#)$, $\pi^{-1}(\pi(D))=D^\#$ and from (i) $D^\#$ is open, hence π is open.

Proof of Lemma 2.3. Assume that E is compact. Note that the set $R := \{(x,y)\in E^m\times E^m \mid \widetilde{x}=\widetilde{y}\}$ is closed. Indeed, let $(x^k,y^k)_k$ be a sequence from R converging to (x,y) in $E^m\times E^m$, hence for all $i\in\{1,\dots,m\}$ we have $x_i^k\longrightarrow x_i$ and $y_i^k\longrightarrow y_i$. For each k let $\tau^k\in\sigma^m$ be such that $y_i^k = x^k_{\tau^k(i)}$ for all $i\in\{1,\dots,m\}$. Since σ^m is a finite set, there exist $\tau^0\in\sigma^m$ and an increasing sequence $(k_p)_p$, $k_p\longrightarrow\infty$, such that $\tau^{k_p}=\tau^0$ for all $p\in\mathbb{N}$. Therefore $y_i^{k_p} = x^{k_p}_{\tau^{k_p}(i)} = x^{k_p}_{\tau^0(i)}\longrightarrow x_{\tau^0(i)}$, $y_i = x_{\tau^0(i)}$ for all $i\in\{1,\dots,m\}$, hence $(x,y)\in R$. We can show now that $E^{(m)}$ is a Hausdorff topological space. Let $\widetilde{x_1},\widetilde{x_2}\in E^{(m)}$ with $x_1,x_2\in E^m$. If $\widetilde{x_1}\neq\widetilde{x_2}$ then $(x_1,x_2)\notin R$. By the first part of the proof we get that (x_1,x_2) belongs to the open set R^c. Consider an open neighborhood $(U,V)\subset R^c$ of (x_1,x_2). By Remark A.2.1(ii) it follows that $\pi(U)$ and $\pi(V)$ are disjoint neighborhoods of $\widetilde{x_1}$ and $\widetilde{x_2}$ respectively. Because π is continuous and E^m is compact metrizable we conclude that $E^{(m)}$ has the same property.

We prove that

$$\pi\big(\mathcal{B}^m\big) = \mathcal{B}^{(m)}. \tag{A.1}$$

The inclusion $\pi(\mathcal{B}^m)\supset\mathcal{B}^{(m)}$ is clear since π is continuous and surjective. Let $F\in\mathcal{B}^m$. By Remark A.2.1(i) we get $F^\#\in\mathcal{B}^m$, therefore we may assume that $F=F^\#$ and thus $\pi(F^c)=\pi(F)^c$. Because π is continuous, we deduce by Lusin's Theorem (cf., e.g., [23]) that $\pi(F)$ and $\pi(F)^c$ are both analytic subsets of the compact metrizable space $E^{(m)}$, therefore $\pi(F)$ is a Borel subset of $E^{(m)}$.

If E is a Radon topological space we consider a compact metrizable space K such that $E\in\mathcal{B}^u(K)$ and $\mathcal{T}=\mathcal{T}(K)|_E$. Again by Remark A.2.1(ii) we have $\mathcal{T}(E^{(m)})=\mathcal{T}(K^{(m)})|_{E^{(m)}}$. If in addition $E\in\mathcal{B}(K)$ then $E^m\in\mathcal{B}(K^m)$ and by (A.1) $E^{(m)} = \pi(E^m)\in\mathcal{B}(K^{(m)})$. We conclude that $(E^{(m)},\mathcal{T}^{(m)})$ is a Lusin topological space, provided that $(E,\mathcal{T})$ is Lusin. In order to show that $(E^{(m)},\mathcal{T}^{(m)})$ is a Radon topological space, it remains to prove that $E^{(m)}\in\mathcal{B}^u(K^{(m)})$. Let μ be a finite measure on $(K^{(m)},\mathcal{B}(K^{(m)}))$. As in the proof of Proposition 4.2 we consider the measure μ_0 on $(K^m,\mathcal{B}(K^m))$ defined as $\mu_0(M) := \mu(\pi(M^\#))$, $M\in\mathcal{B}(K^m)$. Since $E^m\in\mathcal{B}^u(K^m)$, there exist $E',E''\in\mathcal{B}(K^m)$ such that $E'\subset E^m\subset E''$ and $\mu_0(E''\setminus E')=0$. We have $(E''\setminus E')^\#\supset E''^\#\setminus E'^\#$ and $E'^\#\subset E^m\subset E''^\#$. Hence $\mu(\pi(E''^\#)\setminus\pi(E'^\#))\leqslant\mu(\pi((E''\setminus E')^\#)) = \mu_0(E''\setminus E')=0$. $\square$

Proof of Lemma 2.4. If $f\in p\mathcal{B}^m$ is taking finite many values then by (A.1) $\widetilde{f}\in p\mathcal{B}^{(m)}$. Assertion (i) follows now by approximation.

(ii) Suppose that E is compact. By Lemma 0.2 from [15] the linear hull of $\widetilde{\mathcal{C}(E)^{\otimes m}}$ is dense in $\mathcal{C}(E^{(m)})$ and so, using also (i), $\mathcal{B}^{(m)}\supset\sigma(\widetilde{(bp\mathcal{B})^{\otimes m}})\supset\sigma(\widetilde{\mathcal{C}(E)^{\otimes m}}) = \sigma(\mathcal{C}(E^{(m)}))=\mathcal{B}^{(m)}$. If E is a Radon topological space then let K be a compact space such that $E\in\mathcal{B}^u(K)$ and $\mathcal{T}=\mathcal{T}(K)|_E$. By Lemma 2.3 $E^{(m)}$ is also a Radon topological space, more precisely $E^{(m)}\in\mathcal{B}^u(K^{(m)})$ and $\mathcal{B}^{(m)}=\mathcal{B}(K^{(m)})|_{E^{(m)}}$. Since $bp\mathcal{B}=bp\mathcal{B}(K)|_E$, from the above considerations we get $\sigma(\widetilde{(bp\mathcal{B})^{\otimes m}}) = \sigma(\widetilde{(bp\mathcal{B}(K))^{\otimes m}|_{E^m}}) = \sigma(\widetilde{(bp\mathcal{B}(K))^{\otimes m}})|_{E^{(m)}} = \mathcal{B}(K^{(m)})|_{E^{(m)}}$.

(iii) By (ii) it is sufficient to show that if $m\geqslant 2$ and $f\in(bp\mathcal{B})^{\otimes m}$, $f=f_1\otimes\cdots\otimes f_m$, then $f^\#$ belongs to the linear hull of $\mathcal{A}_0 := \{\underbrace{h\otimes\cdots\otimes h}_{m\text{ times}} \mid h\in p\mathcal{B},\ h\leqslant 1\}$. It turns out that the argument from the proof of Lemma 0.2 in [15] works in this context. Namely, by Theorem 1.2 from [22]

$$\sum_{\tau\in\sigma^m}\prod_{i=1}^m f_{\tau(i)}(x_i) = \prod_{i=1}^m\left(\sum_{k=1}^m f_k(x_i)\right) - \sum_{(k_1,\dots,k_{m-1})}\prod_{i=1}^m\left(\sum_{q=1}^{m-1} f_{k_q}(x_i)\right)$$

$$+\sum_{(k_1,\ldots,k_{m-2})}\prod_{i=1}^{m}\left(\sum_{q=1}^{m-2} f_{k_q}(x_i)\right) - \cdots + (-1)^m \sum_k \prod_{i=1}^{m} f_k(x_i),$$

where $\sum_{(k_1,\ldots,k_r)}$ denotes the sum over all $(k_1,\ldots,k_r)$ such that $1 \leqslant k_i \leqslant m$ and all k_i are different, $i \in \{1,\ldots,r\}$, $r \in \mathbb{N}$. Because $f^{\#}(x) = \frac{1}{m!}\sum_{\pi\in\sigma^m}\prod_{i=1}^{m} f_{\pi(i)}(x_i)$, we conclude that $f^{\#}$ belongs to the linear hull of $\mathcal{A}_0$.

(iv) Let μ_1, μ_2 be two finite measures on S such that $\mu_1(g) = \mu_2(g)$ for all $g \in \mathcal{A}$. Note that the family $\mathcal{A}$ is a multiplicative class of bounded functions on S and by assertion (iii) we have $\sigma(\mathcal{A}) = \mathcal{B}(S)$. Using a monotone class argument (see [23, Theorem A0.6]), we conclude that $\mu_1 = \mu_2$. $\square$

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